No. XXVIII.

Research concerning the Mean Diameter of the Earth. By R. Adrain.—Read, Nov. 7, 1817.

THE figure of the earth approaches nearly to that of an oblate spheroid of revolution, the axis being to the equatorial diameter in the ratio of 320 to 321. When this figure is made use of in navigation, geography, &c. the calculations become much more abstruse and laborious than when we consider the earth simply as a sphere. In certain cases, where extreme accuracy is necessary, the oblate figure must be taken into account; but in general, the globular figure will still be retained, as sufficiently accurate for most purposes, of great simplicity in theory, and of easy calculation in practice.

But, if we substitute a sphere instead of the spheroid with which the figure of the earth very nearly coincides, we are by no means at liberty to choose the diameter of the sphere without restriction: we must select a sphere agreeing with the spheroid in as many important circumstances as possible. Of these the following deserve particular attention.

I. The sphere should be equal in magnitude to the spheroid.

II. The mass of the sphere should be equal to that of the spheroid.

III. The surface of the sphere should be equal to the surface of the spheroid.

IV. The length of a degree of a great circle on the surface of the sphere should be a mean of all the degrees of great ellipses on the surface of the spheroid.

V. The radius of the sphere should be a mean of all the radii of the spheroid.

VI. The gravity on the surface of the sphere should be equal to the mean gravity on the surface of the spheroid.

When the spheroid differs very little from a sphere, as in the present case, so that we may neglect, as inconsiderable, all the powers of the ellipticity above the first, we are led to a remarkable coincidence; for all these conditions are fulfilled by one and the same sphere. The determination of this sphere is the object of the following calculations.

PROBLEM I.

To determine the radius of a sphere equal in magnitude to a given oblate spheroid of small ellipticity.

SOLUTION.

Let a and b be the greater and less semiaxes of the spheroid, $r=$ the radius of the required sphere, and $\pi=$ the circumference of a circle to the diameter unity.

By mensuration, the magnitude of the spheroid is $\frac{4\pi}{3}.a^2b$, and that of the sphere is $\frac{4\pi}{3}.r^3$; we have therefore $\frac{4\pi}{3}.r^3=\frac{4\pi}{3}.a^2b$, consequently $r^3=a^2b$.

It is evident, therefore, that the radius r is the first of two mean proportionals between a and b.

When a and b are nearly in the ratio of equality, let $b=a-c$, and by substitution $r^3=a^3-a^2c$, of which the cube root, retaining only the first power of c, is $r=a-\frac{1}{3}c$;

or which is the same thing, $r=b+\frac{2}{3}c$, or $r=\frac{2a+b}{3}$.

According to these formulæ r is the first of two arithmetical means between a and b, and may be found by taking from the greater one third of the difference, or by adding to the less two-thirds of the same difference.

When the less semiaxis b is denoted by unity, and the greater by $1+\delta$, the ellipticity being δ, we have $r+1=\frac{2}{3}\delta$.

We may easily determine in what latitude the semidiameter of the spheroid is equal to the radius of the equivalent sphere.

For this purpose, let ρ = radius or semidiameter of the spheroid in the latitude λ; and when only the first power of δ is retained, we have, by the nature of the ellipse,

$$\rho=1+\delta\cos^2\lambda.$$

And since $r=\rho$, we have

$$1+\delta\cos^2\lambda=1+\frac{2}{3}\delta,$$

therefore $\cos^2\lambda=\frac{2}{3}$, or $\sin^2\lambda=\frac{1}{3}$; consequently the latitude $\lambda=35°\ 16'$, in which the semidiameter of the spheroid is equal to the radius of a sphere equal in magnitude to the spheroid.

When the densities of the sphere and spheroid are uniform and equal, it follows from this proposition that their masses are equal, when $r=1+\frac{2}{3}\delta$.

PROBLEM II.

To determine the radius of a sphere, of which the mass may be equal to that of a given oblate spheroid of small ellipticity, when the density is variable.

SOLUTION.

Conceive the spheroid to be divided by concentric spheroidal surfaces into an infinite number of similar orbs having their axes proportional to the axes of the whole spheroid; and suppose the density in each orb to be uniform, but variable from one orb to another, according to any law whatever. Draw, from the center of the spheroid, a radius to the parallel in which the square of the sine of the latitude is $\frac{1}{3}$; with the several distances from the center to the points in which this radius cuts the surfaces of the orbs, describe spherical surfaces, comprehending an infinite number of spherical orbs; and suppose the density in each spherical orb to be the same with the density in the corresponding spheroidal orb.

It evidently follows from the preceding solution, that the magnitudes of the corresponding spherical and spheroidal orbs are equal; because these orbs are the differences of spheroids and of corresponding spheres respectively equal to them. And since by supposition the density is the same in two corresponding spherical and spheroidal orbs, the masses in these orbs are equal, and therefore the sum of the masses of all the orbs in the sphere is equal to the sum of the masses of all the orbs in the spheroid; that is, the mass of the sphere is equal to the mass of the spheroid: the radius of the sphere being equal to the radius of the spheroid in latitude 35° 16′. We have therefore, $r=1+\frac{2}{3}\delta$, where $r=$ the radius of the required sphere as in the preceding problem.

PROBLEM III.

To determine the radius of a sphere, of which the surface may be equal to that of a given oblate spheroid of small ellipticity.

SOLUTION.

Retaining the preceding notation, let x be an abscissa reckoning on the less axis from the center, and y the corresponding rectangular coordinate of the elliptic meridian, and by the property of the ellipse the equation of the meridian is

$$a^2x^2+b^2y^2=a^2b^2.$$

From this equation we easily find the differential of the spheroidal surface to be

$$\frac{2\pi a}{b}.dx\sqrt{b^2+\frac{a^2-b^2}{b^2}.x^2},$$

the integral of which may be given in general terms by logarithms; but in our problem the ellipticity is supposed to be very small: we may therefore proceed as follows.

Put $b=1$, $a=1+\delta$, and we have $a^2-b^2=2\delta$, the powers of δ above the first being neglected; the preceding differential thus becomes $2\pi(1+\delta).dx\sqrt{1+2\delta x^2}$.

But $\sqrt{1+2\delta x^2}=1+\delta x^2$, and therefore the differential of the surface becomes $2\pi dx(1+\delta+\delta x^2)$,
of which the integral beginning with x is

$$2\pi\left\{(1+\delta)x+\frac{\delta x^3}{3}\right\}.$$

When $x=1$ we have half the surface $=2\pi(1+\frac{4\delta}{3})$, therefore, the whole surface of the spheroid is $4\pi(1+\frac{4\delta}{3})$.

Again, the surface of a sphere to the radius r is $4\pi r^2$, we must therefore have $4\pi r^2 = 4\pi(1+\frac{4\delta}{3})$;

whence $r^2 = 1+\frac{4\delta}{3}$, and consequently $r = 1+\frac{2\delta}{3}$.

This result gives precisely the same value of r as in the preceding problems; it is manifest, therefore, that a sphere of which the radius is equal to the semidiameter of the spheroid in latitude 35° 16′ will have its surface and solidity respectively equal to those of the spheroid.

The surface of the spheroid of small ellipticity may be easily determined in a different manner, as follows.

Let λ denote the reduced latitude, or as it is called by Delambre, the geocentric latitude, which is the angle contained betwen ϱ and a; and by conics, retaining only the first power of δ, we have

$$\varrho = 1+\delta\cos^2\lambda;$$

the value of ϱ being the same in terms of λ as when λ is the common latitude, the powers of δ above the first being rejected. Also ϱ may be considered as at right angles to the meridian, because the sine of the angle which ϱ makes with the true elliptic meridian does not involve δ, but δ^2, δ^3, &c.

In this case the element of the meridian will be denoted by $\varrho.d\lambda$, which multiplied by the length of the parallel $2\pi\varrho\cos\lambda$, gives $2\pi\varrho^2.d\lambda\cos\lambda$ for the differential of the spheroidal surface. But $\varrho^2 = 1+2\delta\cos^2\lambda$, therefore the differential of the surface is

$$2\pi(d\lambda\cos\lambda + 2\delta.d\lambda\cos^3\lambda),$$

of which the integral beginning with λ is

$$2\pi\left\{\sin\lambda+\delta(2\sin\lambda-\frac{2}{3}\sin^3\lambda)\right\}.$$

This expression when $\lambda=\frac{\pi}{2}$ becomes $2\pi(1+\frac{4\delta}{3})$, and therefore the whole surface $=4\pi(1+\frac{4\delta}{3})$, which coincides exactly with the result of the preceding investigation.

PROBLEM IV.

To find a sphere, on which the degree of a great circle may be a mean of all the degrees of great ellipses of a given oblate spheroid of small ellipticity.

SOLUTION.

Since the degree is always proportional to the radius of curvature, it is obvious that the proposed problem is equivalent to the following:

To find a sphere, of which the radius may be a mean of all the radii of curvature of the spheroid.

The semiaxes of the spheroid being denoted by 1 and $1+\delta$, and the latitude by λ; let R be the radius of curvature of the meridian in latitude λ, and R' the radius of curvature of the central ellipse at the point where it cuts the meridian at right angles in the same latitude.

By conics and the differential calculus we have

$$R=1+\delta(2-3\cos^2\lambda),\text{ and } R'=1+\delta(2-\cos^2\lambda);$$

the latter R' being the same with the radius of curvature of the vertical section cutting the meridian at right angles in latitude λ, when we neglect the powers of δ above the first.

Again, let A be the angle which a vertical or central ellipse makes with the meridian in latitude λ, and R'' the radius of curvature of this section in the same latitude; and by the differential calculus we have $R''=R+(R'-R)\sin^2 A$.

Multiply this equation by dA, and we have

$$R''dA=RdA+(R'-R).dA\sin^2 A,$$

which is the measure of the sum of all the radii of curvature in the angle dA: and the integral beginning with A is

$$RA+(R'-R).\left(\frac{A}{2}-\frac{1}{4}\sin 2A\right)$$

expressing the measure of the sum of all the radii of curvature in the angle A. When $A=2\pi=$ a circumference, this sum becomes $\pi.(R+R')$, which, by putting for R and R′ their values given above in terms of δ and λ, is expressed by

$$2\pi\{1+2\delta\sin^2\lambda\}:$$

and this is the proper measure of the sum of all the radii of curvature at any point of the meridian.

Now in any equal particles of surface it is evident that an equal number of radii of curvature may be drawn; we must therefore multiply the preceding sum

$$2\pi\{1+2\delta\sin^2\lambda\}$$

by the differential of the spheroidal surface

$$2\pi.d\lambda\cos\lambda\{1+2\delta\cos^2\lambda\},$$

and the product $4\pi^2.(1+2\delta).d\lambda\cos\lambda$ is the differential of the sum of all the radii of curvature. The integral beginning with λ is $4\pi^2.(1+2\delta)\sin\lambda$, which, when $\lambda=\frac{\pi}{2}$, is $4\pi^2.(1+2\delta)$: and the double of this, viz. $8\pi^2.(1+2\delta)$ is the measure of the sum of all the radii of curvature on the surface of the spheroid.

By reasoning in a similar manner with a sphere to the radius r, we have the sum of all the radii at any point $=2\pi r$, which multiplied by the differential of the spherical surface $2\pi r^2.d\lambda\cos\lambda$ gives $4\pi^2r^3.d\lambda\cos\lambda$ for the differential of the sum of the radii of the sphere. The integral beginning with λ is $4\pi^2r^3.\sin^2\lambda$; which, by making $\lambda=\frac{\pi}{2}$, and then doubling, becomes $8\pi^2.r^3$ for the measure of the sum of all the radii in the sphere.

Lastly, when the radius of the sphere is a mean of all the radii of curvature of the spheroid, the two integrals found above must be equal; we have therefore, $8\pi^2.r^3=8\pi^2.(1+2\delta)$, whence $r^3=1+2\delta$, and consequently $r=1+\frac{2}{3}\delta$, which is the radius of the required sphere, and agrees exactly with what has been found in the solutions of the preceding problems.

Nearly related to this problem is the following, the solution of which being nearly similar to that just exhibited, need not be given in detail.

PROBLEM V.

To find a sphere of which the curvature may be equal to the mean curvature of a given oblate spheroid of small ellipticity.

SOLUTION.

The curvature being inversely as the radius of curvature, we have only to use every where the reciprocals of R, R′, R″, instead of these quantities themselves in the preceding solution. The curvature of the meridian and of the great ellipse at right angles to it will be measured by $1-\delta.(2-3\cos^2\lambda)$, and $1-\delta.(2-\cos^2\lambda)$; and the value of the required radius of the sphere is found as before $r=1+\frac{2}{3}\delta$.

PROBLEM VI.

To find a sphere of which the *n*th power of the radius may be a mean of the *n*th powers of all the radii of an oblate spheroid of small ellipticity.

SOLUTION.

The solid angle at the center of the spheroid, or the corresponding spherical surface to the radius unity, is the measure of the number of radii that can be drawn to the corresponding elementary particle of surface on the spheroid. This solid angle or spherical surface is expressed by $2\pi.d\lambda\cos\lambda$, which, multiplied by ϱ^n gives $2\pi\varrho^n.d\lambda\cos\lambda$ for the differential of the sum of nth powers of the radii in the spheroid.

But since $\varrho=1+\delta.\cos\cdot\lambda$, therefore $\varrho^n=1+n\delta\cos^2\lambda$, and therefore the differential just found, becomes

$$2\pi\left\{d\lambda\cos\lambda+n\delta d\lambda\cos^3\lambda\right\}.$$

The integral of this, beginning with λ, is

$$2\pi\left\{\sin\lambda+n\delta\left(\sin\lambda-\frac{1}{3}\sin^3\lambda\right)\right\},$$

which, when $\lambda=\frac{\pi}{2}$ becomes $2\pi\left\{1+\frac{2n\delta}{3}\right\}$; and the double of this, viz. $4\pi\left\{1+\frac{2n\delta}{3}\right\}$, is the sum in the case of the spheroid.

Again, r being the radius of the sphere, we obtain by a similar method $4\pi.r^n$ for the sum of the nth powers of the radii: and since, when r^n is a mean of all the ϱ^n the latter sum must be equal to the former; we have therefore

$$4\pi r^n=4\pi.\left\{1+\frac{2n\delta}{3}\right\},$$

whence $r^n=1+\frac{2n\delta}{3}$, and extracting the nth root, we have $r=1+\frac{2}{3}\delta$; the same result as in the preceding problems.

In like manner we may find the radius of the sphere such that its nth power may be a mean of the nth powers of all the radii of curvature in the spheroid: and the result will be the same as before.

PROBLEM VII.

To find a sphere such that any function of its radius may be a mean of the similar functions of all the radii of an oblate spheroid of small ellipticity.

SOLUTION.

Let $\varphi(\varrho)$ be any function of the radius ϱ, and according to what was shown in problem 6th, the differential of the sum of all the $\varphi(\varrho)$ in the spheroid is $2\pi.\varphi(\varrho).d\lambda\cos\lambda$, or which is the same thing, $2\pi d\lambda\cos\lambda.\varphi(1+\delta\cos^2\lambda)$.

But $\varphi(1+\delta\cos^2\lambda)=A+B\delta\cos^2\lambda$, in which A and B are numbers deduced from unity according to the form of the function φ: the preceding differential therefore becomes

$$2\pi.\{Ad\lambda\cos\lambda+B\delta.d\lambda\cos^3\lambda\},$$

of which the integral beginning with λ is

$$2\varphi.\left\{A\sin\lambda+B\delta(\sin\lambda-\frac{1}{3}\sin^3\lambda)\right\}.$$

This integral, when $\lambda=\frac{\pi}{2}$, becomes $2\pi.\left\{A+\frac{2}{3}B\delta\right\}$, the dou- of which $4\pi.\left\{A+\frac{2}{3}B\delta\right\}$ is the measure of the sum of all the similar functions of ϱ in the spheroid.

Again, let us denote the required radius r by $1+a$, and the differential of the sum in the sphere is $2\pi d\lambda\cos\lambda.\varphi(1+a)$.

But since $\varphi(1+\delta\cos^2\lambda)=A+B.\delta\cos^2\lambda$, therefore $\varphi(1+a)=A+Ba$, as is evident by writing a instead of $\delta\cos^2\lambda$, and A and B are the same numbers for the sphere as for the spheroid: the differential of the sum in the sphere is therefore $2\pi d\lambda\cos\lambda.(A+Ba)$, of which the integral, beginning with λ, is $2\pi\sin\lambda.(A+Ba)$; and by putting $\lambda=\frac{\pi}{2}$, and doubling the result, we have the sum of all the similar functions of the radii in the sphere expressed by $4\pi.(A+Ba)$.

Now this sum must be equal to that found in the case of the spheroid; we have therefore

$$4\pi.(A+Ba)=4\pi.(A+\frac{2}{3}B\delta),$$

whence $A+Ba=A+\frac{2}{3}B\delta$, and consequently $a=\frac{2}{3}\delta$, and therefore $1+a=r=1+\frac{2}{3}\delta$ as before.

By a method nearly similar, we may resolve the following problem. To find a sphere such that any function of its radius may be a mean of the similar functions of all the radii of curvature in the spheroid.

Each of the last two problems comprehends the other, and all those in the preceding part of this paper, with many others which it is unnecessary to mention: we have therefore good reason to conclude that the mean diameter of the earth is truly determined by the formula $r=a-\frac{a-b}{3}$, or which amounts to the same thing $r=b+\frac{2}{3}(a-b)$, or $r=\frac{2a+b}{3}$, or more simply by $r=1+\frac{2}{3}\delta$.

PROBLEM VIII.

To determine the gravity which ought to be assigned to the earth's surface when taken as a sphere.

SOLUTION.

Let g and g' be the gravities at the pole and equator of the terrestrial spheroid, and, by the theory of gravity on the surface of revolving spheroids, the gravity in latitude λ is $g'+(g-g')\cos^2\lambda$, which, multiplied by the differential of the surface $2\pi\varrho^2.d\lambda\cos\lambda$, gives

$$2\pi d\lambda\cos\lambda\,(1+2\delta\cos^2\lambda)\,.\,(g'+(g-g')\cos^2\lambda),$$

or

$$2\pi\left\{g'd\lambda\cos\lambda+(g-g'+2g'\delta).d\lambda\cos^3\lambda\right\}$$

for the differential of the measure of the whole gravity on every part of the surface of the earth. This differential integrated so as to begin with λ gives

$$2\pi \left\{ g' \sin\lambda + (g - g' + 2g'\delta).(\sin\lambda - \frac{1}{3}\sin^3\lambda) \right\};$$

which by taking $\lambda = \frac{\pi}{2}$, and doubling the result, gives

$$4\pi. \left\{ g' + \frac{2}{3}(g - g' + 2g'\delta) \right\}$$

for the measure of the whole gravitation on the surface of the spheroid.

This quantity divided by the whole surface of the spheroid, or of the sphere having an equal surface, viz. by

$$4\pi.(1 + \frac{4}{3}\delta),$$

the quotient $g' + \frac{2}{3}(g - g')$, or $\frac{2g + g'}{3}$, is the mean gravity required.

It is easy to perceive that this gravity also belongs to the latitude 35° 16′ in which $\cos^2\lambda = \frac{2}{3}$, as in the determination of the mean radius r.

In this latitude 35° 16′ in which the surface of the mean sphere cuts the surface of the terrestrial spheroid, the attraction towards the sphere is equal to the attraction towards the spheroid, whether we suppose the densities of both to be uniform, or to vary according to the law adopted in the solution of the second problem, when the powers of δ above the first are neglected. We may conclude, therefore, that the radius of the required mean sphere and the gravity on its surface should be equal to the semidiameter and gravity of the terrestrial spheroid in latitude 35° 16′.

Having determined the most probable axes of the terrestrial spheroid from the measurement of degrees of the meridian by a method which I discovered several years ago, and published in *The Analyst;* the resulting mean radius was found to be 3959.36 English miles. The diameter of the earth taken as a sphere is therefore 7918.7, the circumference 24877.4, and the length of a degree of a great circle 69.104, or $69\frac{1}{10}$ English miles very nearly.